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# COMBINED SPACE AND TIME CONVERGENCE ANALYSIS OF A COMPRESSIBLE FLOW ALGORITHM

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## Abstract

We describe an analysis of the combined spatial and temporal convergence of a compressible flow algorithm. Unlike other analyses, in this study we examine the space-dependent and time-dependent aspects *together*. This analysis includes the numerical solution of a set of nonlinear equations that model discretization error. The unknowns in these equations are parameters, including the asymptotic convergence rates, that provide metrics used to quantify the performance of the software implementation of the algorithm. These measures gauge the difference between the computed and exact solutions. Restricted to a smooth problem, the design accuracy of the algorithms should be achieved. While we focus on the Euler equations of gasdynamics in this note, the verification analysis presented contains the elementary concepts in sufficient detail to apply this technique to a variety of different algorithms.

## Introduction

In this study, we quantify both the spatial and temporal convergence behavior *simultaneously* of an algorithm for the two-dimensional Euler equations of gasdynamics. Such an analysis falls under the rubric of verification, which is the process of determining whether a simulation code accurately represents the code developers' description of the model (e.g., equations, boundary conditions, etc.).[1] The recognition that verification analysis is a necessary and valuable activity continues to increase among computational fluid dynamics practitioners.[2,3]

Using computed results and a known solution, one can estimate the effective convergence rates of a specific software implementation of a given algorithm and gauge those results relative to the design properties of the algorithm. In the aerodynamics community, such analyses are typically undertaken to evaluate the performance of spatial integrators; analogous convergence analysis for temporal integrators can also be conducted. Our approach combines these two usually separate activities into the same analysis framework.

To accomplish this task, we outline a procedure in which a known solution together with a set of computed results, obtained for a number of different spatial and temporal discretizations, are employed to determine the complete convergence properties of the combined spatio-temporal algorithm. Such an approach is of particular interest for Lax-Wendroff-type integration schemes, where the specific impact of either the spatial or temporal integrators alone cannot be easily deconvolved from computed results. Unlike the more common spatial convergence analysis, the combined spatial and temporal analysis leads to a set of nonlinear equations that must be solved numerically. The unknowns in this set of equations are various parameters, including the asymptotic convergence rates, that quantify the basic performance of the software implementation of the algorithm.

Theoretical results for convergence properties of algorithms for the Euler equations are most frequently obtained for smooth problems. Therefore, we present preliminary results for simultaneous spatio-temporal convergence properties of two-dimensional smooth problems involving linear fields of the equations. An obvious extension of this effort would include verification analysis of smooth problems that involve the nonlinear fields.[4] The novel verification procedure described herein can also be applied to problems that develop discontinuous solutions (involving, e.g., shockwaves[5]), for which there exist some theoretical convergence results.[6]

This paper continues with a brief description of the general error ansatz comprised of both spatial and temporal discretization errors. The solution of the restricted convergence ansatz for a single independent variable is then discussed, followed by a description of the solution of the coupled space-time case. To demonstrate this approach, the combined convergence

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analysis for a simple advection problem is presented. A conclusion summarizes the contents and results of this paper.

## General Error Ansatz

We consider the partial differential equation $\mathcal{N}(U) = 0$, where $\mathcal{N}$ is a nonlinear operator consisting of spatial and temporal partial derivatives and $U$ is a real-valued solution. We are concerned with quantifying the accuracy of numerical solutions to this equation. To characterize the combined space and time dependence of the error in the solution, we analyze the *average* per-timestep convergence properties by proposing the following error ansatz:

$$\begin{aligned} ||\xi^* - \xi_i^l||/N_{\Delta t_l} = \mathcal{E}_0 + \mathcal{A}\ (\Delta x_i)^p + \mathcal{B}\ (\Delta t_l)^q \\ + \mathcal{C}\ (\Delta x_i)^r\ (\Delta t_l)^s \\ + o\Big((\Delta x_i)^p, (\Delta t_l)^q, (\Delta x_i)^r\ (\Delta t_l)^s\Big), \end{aligned} \quad (1)$$

where $\xi$ is some functional of the solution (e.g, one component of $U$), $\xi^*$ is the *exact* value, $\xi_i^l$ is the value computed on the grid of spatial zone size $\Delta x_i$ with timestep $\Delta t_l$, $||\cdot||$ is a norm that maps its argument to the non-negative real numbers,[7] $N_{\Delta t_l}$ is the number of time cycles taken to obtain the solution at the final time, $\mathcal{E}_0$ is the *zeroth-order error*, $\mathcal{A}$ is the *spatial convergence coefficient*, $p$ is the *spatial convergence rate*, $\mathcal{B}$ is the *temporal convergence coefficient*, $q$ is the *temporal convergence rate*, $\mathcal{C}$ is the *spatio-temporal convergence coefficient*, and $r+s$ is the *spatio-temporal convergence rate.*

The relation in Eq. 1 averages out the position-to-position, cycle-to-cyle dependence of the computed results on $\Delta x$ and $\Delta t$. In this expression, the solution norm, which is typically a discrete approximation to some integral of its argument, can be interpreted as a spatial averaging operator; that is, the norm quantifies some mean measure of the spatial behavior of its argument. The ratio of this quantity with the number of computational cycles is effectively a temporal averaging operator; in analogy with the spatial norm, this operation produces a mean per-timestep measure.

We make two additional assumptions: (i) we assume that the zeroth-order error is negligible, i.e.,

$$|\,\mathcal{E}_0\,| \ll |\,\mathcal{A}\ (\Delta x_i)^p\,|, |\,\mathcal{B}\ (\Delta t_l)^q\,|, |\,\mathcal{C}\ (\Delta x_i)^r\ (\Delta t_l)^s\,|; \quad (2)$$

and (ii) we presume that we can evaluate *a priori* the exact solution $\xi^*$ at any spatial position at a specified time.

### PDE Solver Discretization Errors

Features of the numerical method used to convert continuous PDEs into a set of discrete equations affect the nature of the discretization error associated with the computational solution. For example, numerical solution methods may produce either independent or interrelated space and time discretization errors, as modeled in the error ansatz. Two different methods may have different convergence rates or, alternatively, the same convergence rates but different absolute errors. Additionally, the overall efficiency may vary significantly between two schemes that have comparable convergence rates.[8]

Modern high-resolution numerical schemes for hyperbolic conservation laws[9,10] may not retain strict separation of spatial and temporal discretizations. Therefore, for such methods the interaction of the spatial and temporal discretization errors may occur. Consequently, examination of combined space and time convergence analysis provides the most accute insight into the performance of, e.g., certain compressible flow algorithms for ideal gasdynamics.

As suggested by this observation, the selection of a restricted or a combined verification approach should be based on the functional relationships contained in the underlying numerical method being analyzed. Combined verification is indicated when the errors associated with the spatial and temporal discretizations are mathematically coupled in the numerical method being used to solve the underlying equations. Restricted verification analysis, on the other hand, is appropriate when the spatial and temporal discretizations in the numerical method are mathematically independent. A combined convergence analysis could be employed in this case; however, such an approach presents an unnecessarily complicated methodology for characterizing independent discretization errors.

### Restricted Convergence Analysis

There are several special cases of Eq. 1, depending upon the relative magnitude of the terms in this equation. If the spatial error dominates the temporal error, i.e., if

$$|\,\mathcal{A}\ (\Delta x_i)^p\,| \gg |\,\mathcal{B}\ (\Delta t_l)^q\,|, |\,\mathcal{C}\ (\Delta x_i)^r\ (\Delta t_l)^s\,|, \quad (3)$$

then a pure spatial convergence analysis can be conducted. Alternatively, if the temporal error dominates the spatial error, i.e., if

$$|\,\mathcal{B}\ (\Delta t_l)^q\,| \gg |\,\mathcal{A}\ (\Delta x_i)^p\,|, |\,\mathcal{C}\ (\Delta x_i)^r\ (\Delta t_l)^s\,|, \quad (4)$$

then a pure temporal convergence analysis can be conducted.

In either of these cases, the technique by which to determine the convergence characteristics (i.e., the values of the parameters $\mathcal{A}$, $\mathcal{B}$, $p$, and $q$) is straightforward; by way of example, we consider the one-dimensional spatial convergence case (the temporal convergence case is similar). In this case, we suppress the temporal index $l$, subsume the division by the number of timesteps into the norm, and obtain the following relation:

$$||\xi^* - \xi_i|| = \mathcal{A}\ (\Delta x_i)^p + \cdots . \quad (5)$$

For the solution on a "coarse" grid, denoted with the subscript $c$, $\Delta x_c \equiv \Delta x$, and Eq. 5 implies

$$||\xi^* - \xi_c|| = \mathcal{A}\,(\Delta x)^p + \cdots \,. \tag{6}$$

Consider now the computed solution on a "fine" grid with cell size $\Delta x_f$ such that $\Delta x_c/\Delta x_f = \Delta x/\Delta x_f = \sigma > 1$, i.e., $\Delta x_f = \Delta x_c/\sigma = \Delta x/\sigma < \Delta x$. For this solution, Eq. 5 implies

$$||\xi^* - \xi_f|| = \sigma^{-p}\,\mathcal{A}\,(\Delta x)^p + \cdots \,. \tag{7}$$

Equations 6 and 7 form a system of two equations in the two unknowns $\mathcal{A}$ and $p$. This system can be solved explicitly for these quantities:

$$\begin{aligned} p &\equiv \left[\log(||\xi^* - \xi_c||) - \log(||\xi^* - \xi_f||)\right] / \log\sigma \,, \\ \mathcal{A} &\equiv \left(\,||\xi^* - \xi_c||\,\right) / (\Delta x)^p \,. \end{aligned} \tag{8}$$

The convergence rate $p$ and convergence coefficient $\mathcal{A}$ constitute *metrics* that gauge the code's convergence properties. For numerical methods with known convergence properties, these values can be used to develop evidence that the solution algorithm has been properly implemented in the software.

**Combined Convergence Analysis**

The general error ansatz in Eq. 1 contains a total of seven unknowns: $A$, $p$, $B$, $q$, $C$, $r$, and $s$, following the assumption in Eq. 2. Unlike the simplified case of the previous section, in this case there is no general closed-form solution for these parameters. Therefore, to solve for these quantities we require seven independent equations. To do so, we obtain computed solutions at the same final time with the following seven combinations of spatial and temporal zoning:

$$\begin{aligned} (1) &: \{\Delta x \quad , \Delta t\}\,, \\ (2) &: \{\Delta x/\sigma \ , \Delta t\}\,, \\ (3) &: \{\Delta x/\sigma^2, \Delta t\}\,, \\ (4) &: \{\Delta x \quad , \Delta t/\tau\}\,, \\ (5) &: \{\Delta x \quad , \Delta t/\tau^2\}\,, \\ (6) &: \{\Delta x/\sigma \ , \Delta t/\tau\}\,, \\ (7) &: \{\Delta x/\sigma \ , \Delta t/\tau^2\}\,, \end{aligned} \tag{9}$$

where $\sigma > 1$ is the ratio of the spatial grid sizes, and $\tau > 1$ is the ratio of the temporal grid sizes. This set of zonings is neither unique nor demonstrably optimal for obtaining solutions for the parameters in Eq. 1. It does, however, provide a sufficient set of independent information with which to obtain solutions for the unknowns in this equation.

The set of computed solutions on these space-time grids satisfies the following equalities at the (identical) final time:

$$\begin{aligned} 0 = f_1 &= -||\xi^* - \xi_1||/N_c + \mathcal{A}\,(\Delta x_c)^p \\ &\quad + \mathcal{B}\,(\Delta t_c)^q + \mathcal{C}\,(\Delta x_c)^r\,(\Delta t_c)^s\,, \\ 0 = f_2 &= -||\xi^* - \xi_2||/N_c + \mathcal{A}\,(\Delta x_m)^p \\ &\quad + \mathcal{B}\,(\Delta t_c)^q + \mathcal{C}\,(\Delta x_m)^r\,(\Delta t_c)^s\,, \\ 0 = f_3 &= -||\xi^* - \xi_3||/N_c + \mathcal{A}\,(\Delta x_f)^p \\ &\quad + \mathcal{B}\,(\Delta t_c)^q + \mathcal{C}\,(\Delta x_f)^r\,(\Delta t_c)^s\,, \\ 0 = f_4 &= -||\xi^* - \xi_4||/N_m + \mathcal{A}\,(\Delta x_c)^p \\ &\quad + \mathcal{B}\,(\Delta t_m)^q + \mathcal{C}\,(\Delta x_c)^r\,(\Delta t_m)^s\,, \\ 0 = f_5 &= -||\xi^* - \xi_5||/N_f + \mathcal{A}\,(\Delta x_c)^p \\ &\quad + \mathcal{B}\,(\Delta t_f)^q + \mathcal{C}\,(\Delta x_c)^r\,(\Delta t_f)^s\,, \\ 0 = f_6 &= -||\xi^* - \xi_6||/N_m + \mathcal{A}\,(\Delta x_m)^p \\ &\quad + \mathcal{B}\,(\Delta t_m)^q + \mathcal{C}\,(\Delta x_m)^r\,(\Delta t_m)^s\,, \\ 0 = f_7 &= -||\xi^* - \xi_7||/N_f + \mathcal{A}\,(\Delta x_m)^p \\ &\quad + \mathcal{B}\,(\Delta t_f)^q + \mathcal{C}\,(\Delta x_m)^r\,(\Delta t_f)^s\,. \end{aligned} \tag{10}$$

In these expressions, $\Delta x_c \equiv \Delta x$ is the <u>c</u>oarse spatial grid size, $\Delta x_m \equiv \Delta x/\sigma$ is the <u>m</u>edium spatial grid size, and $\Delta x_f \equiv \Delta x/\sigma^2$ is the <u>f</u>ine spatial grid size; similarly, $\Delta t_c \equiv \Delta t$ is the <u>c</u>oarse timestep, $\Delta t_m \equiv \Delta t/\tau$ is the <u>m</u>edium timestep, and $\Delta t_f \equiv \Delta t/\tau^2$ is the <u>f</u>ine timestep. Also, $N_c$, $N_m$, and $N_f$ represent the number of time cycles involved in computing the solutions with the <u>c</u>oarse, <u>m</u>edium, and <u>f</u>ine timesteps, respectively.

Equation 10 can be written as $\mathbf{f}(\mathbf{a}) = \mathbf{0}$, where the elements of $\mathbf{f}$ are indicated above and $\mathbf{a} \equiv [a_1, \ldots, a_7]^\top \equiv [\mathcal{A}, p, \mathcal{B}, q, \mathcal{C}, r, s]^\top$. To obtain solutions to this set of nonlinear equations, we use a modified line-search-based Newton's method.[11] It is a straightforward exercise to derive closed-form expressions for the elements of the corresponding Jacobian $\mathcal{J}$, with elements $\mathcal{J}_{i,j} \equiv \partial f_i/\partial a_j$, the inverse of which is typically evaluated numerically in Newton's method-based routines.

To obtain solutions to Eq. 10, one must obtain the calculated solutions of the underlying PDEs at the fixed final time using the spatial and temporal grids specified. An initial guess must be assigned to the array of unknowns $\mathbf{a}$; this initial guess must be within the domain of convergence of the iterative solution of Eq. 10. The former is a matter of computer resources, whereas the latter requires some *a priori* knowledge of the algorithm of interest. On possible choice for initial guess consists of the algorithm's theoretical convergence rates together with, say, estimates of the convergence coefficients from a purely spatial convergence analysis (as previously discussed).

**Convergence Analysis Requirements**

Restricted (i.e., space- or time-only) and combined (i.e., space and time together) convergence analyses are further distinguished by their requirements. One must be mindful that, in general, the only measurable discretization error is the *total* discretization error, which includes both spatial and temporal contributions. Combined convergence analysis involves the solution of the single ansatz equation that accounts for this situation. A series of simulations for such a study requires a set of numerical calculations using space and time discretizations that are independent, e.g., as described in the previous section. In contrast, a complete restricted convergence analysis requires two separate convergence studies, i.e., one each for space and time independently. In this case, the separation of the two sources of independent discretization errors (i.e., from the spatial and temporal discretizations) can be problematic, due to the possibly unknown magnitudes of the "neglected" component. Combined convergence analysis circumvents such uncertainties inherent to separate convergence analyses.

## The Euler Equations of Gasdynamics

The Euler equations summarize the conservation of mass, momentum, and energy for a compressible fluid. For a single inviscid, compressible fluid, these equations in two-dimensional Cartesian coordinates are:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &+ \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0\,, \\
\frac{\partial(\rho u)}{\partial t} &+ \frac{\partial\left(\rho u^2+p\right)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = 0\,, \\
\frac{\partial(\rho v)}{\partial t} &+ \frac{\partial(\rho u v)}{\partial x} + \frac{\partial\left(\rho v^2+p\right)}{\partial y} = 0\,, \\
\frac{\partial(\rho E)}{\partial t} &+ \frac{\partial\left[\rho u\left(E+\frac{p}{\rho}\right)\right]}{\partial x} + \frac{\partial\left[\rho v\left(E+\frac{p}{\rho}\right)\right]}{\partial y} = 0\,,
\end{aligned}
\tag{11}
$$

where $\rho$ is the mass density, $(u, v)$ are the components of the velocity vector in Cartesian coordinates $(x, y)$, $t$ is the time, $E = e + \frac{1}{2}(u^2 + v^2)$ is the specific total energy, $e$ is the specific internal energy (SIE), and $p = p(\rho, e)$ is the pressure.

To obtain numerical solutions, these continuum equations are approximated on a grid that is discrete in both space and time. Specifically, we consider a fixed Eulerian grid onto which Eq. 11 is discretized. The corresponding solution of the discretized form of Eq. 11 is indicated as $U^l_{i,j}$, where $U \equiv [\rho, \rho u, \rho v, \rho E]^\top$ is the array of conserved variables and $U^l_{i,j}$ corresponds to $U(x_i, y_j; t_l)$, the solution at position $(x_i, y_j)$ and time $t_l$. We assume a *uniform* and *equal* spatial grid with $\Delta x = \Delta y$ and *uniform* and *equal* timesteps $\Delta t$.

The algorithm we consider uses Lax-Wendroff time differencing (a Richymyer-type predictor-corrector[9,12]) together with a Godunov-type method (a high-resolution piecewise linear method[13,14]). Consequently, the temporal dependence is interwoven with the spatial dependence through self-similar solutions to local Riemann problems. Therefore, combined verification analysis, capturing both spatial- and temporal-dependencies, is indicated.

## Combined Convergence Results

Our preliminary results for this technique are based on the evaluation of a smooth problem, i.e., one that possesses smooth initial conditions and that is allowed to evolve to a final time prior to the development of any discontinuities. One way to obtain an exact solution for verification purposes would be to use the Method of Manufactured Solutions.[15–17] Instead, we use known solutions of the Euler equations; our approach obviates the need to deal with extraneous source terms in the equations and possible modifications of the solution algorithm. The numerical solutions that we obtain with different spatial and temporal meshes are compared with the exact solution at identical final times. The convergence properties of the coded algorithm are then inferred following the procedure outlined above.

### Example Problem

The two-dimensional, planar geometry initial conditions for this problem consist of a sinusoidal distribution of density with initially constant and uniform pressure, thermodynamically consistent specific internal energy (SIE), and uniform non-zero velocity $(u_0, v_0)$. The equation of state is chosen to be a polytropic gas with adiabatic exponent $\gamma = 1.4$. With periodic boundary conditions, this configuration advects the sinusoidal density and SIE distributions, which remain unperturbed, through the computational mesh. If we write the initial conditions as $f(x, y)$, then the solution at any time $t$ is given by $f(x - u_0\, t, y - v_0\, t)$. The domain of interest is assigned to be the square of unit dimension centered at the origin in Cartesian geometry, i.e., $\{(x, y) : -1/2 \le x \le 1/2 \text{ and } -1/2 \le y \le 1/2\}$. The initial conditions for this problem are given in Table 1.

One severe limitation of this problem is that it tests only the linear fields in the governing equations. Alternate test problems that exercise the nonlinear fields of Eq. 11 include the smooth simple wave problem proposed by Cabot[4] and nonsmooth 1-D shock tube problems,[5] both of which have exact solutions.

Calculations of the problem considered herein were carried out on uniform grids consisting of $32 \times 32$, $64 \times 64$, $128 \times 128$, and $256 \times 256$ zones. Timesteps of 1/1600, 1/3200, 1/6400, 1/12800 were used; these timesteps are well below the CFL limit for this set of calculations. Thus, both the subsequent spatial and temporal zone sizes used in computing the convergence properties were a factor of two smaller, i.e., $\sigma = \tau = 2$ in the nomenclature used earlier. This choice of $\sigma$ and $\tau$, i.e., of halving the spatial grid size and timestep, was made for convenience only; as discussed, e.g., by Roache,[2] the factors $\sigma$ and $\tau$ are arbitrary positive real numbers, provided the calculated solutions of the underlying equations remain within the range of asymptotic convergence of the numerical method.

It must be emphasized that the solution values must be compared at *identical locations in space* at *exactly the same time*. Our experience, consistent with that suggested by Roache,[2] is that convergence analysis proves to be exquisitely sensitive to minor discrepancies in either the code or the procedure (such as slight

2-D Sinusoidal Density Advection Problem Initial Conditions

| $\gamma$ | $\rho$ | $p$ | e | $u$ | $v$ |
|---|---|---|---|---|---|
| 1.4 | $2+\sin 2\pi x\ \sin 2\pi y$ | 1.0 | $2.5/\left(2+\sin 2\pi x\ \sin 2\pi y\right)$ | 1.0 | 1.0 |

**Table 1 Initial values of the adiabatic exponent $\gamma$, nondimensional density $\rho$, pressure $p$, SIE $e$, $x$-velocity $u$, and $y$-velocity $v$ for the 2-D sinusoidal density advection problem.**

differences in the final simulation time of calculations being compared). Interpolation of solutions provides values at identical spatial locations and the choice of fixed timesteps allows solutions to be obtained at the identical final time, $t = 0.1$.

### Demonstration Results

For the problem described in the previous section, preliminary results, based on the suite of calculations conducted on $32 \times 32$, $64 \times 64$, and $128 \times 128$ grids, are presented in Table 2. The corresponding results based on $64 \times 64$, $128 \times 128$, and $256 \times 256$ grids are given in Table 3. As shown in these tables, the spatial, temporal, and combined spatio-temporal convergence rates (i.e., $p$, $q$, and $r + s$) are each approximately two in all cases. These results are in good agreement with the design characteristics of both the spatial and temporal integrators of the code, which are all nominally second order. This comparison of computed convergence rates to their theoretical values is axiomatic in verification because convergence rates are problem independent. These quantities arise through the conversion of continuum PDEs into discrete equations, with both (i.e., continuum and discrete) possessing solutions for a range of problem setup (i.e., various combinations of initial and boundary conditions).

Tables 2 and 3 also show that the convergence coefficients, i.e., $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, are comparable for the compressible flow algorithm used on this problem. This result indicates that each term in the discretization error model is approximately equally important for the simple linear problem considered. Moreover, the comparability of the convergence coefficients suggests that the discretization error model in Eq. 1 is consistent with the software implementation of the numerical method. In particular, these results indicate that the mixed error term may be necessary to properly characterize the discretization error generated by certain modern flow algorithms.

This examination of convergence coefficients is an important component of this study. Such evaluations are typically not used to gauge the proper implementation of numerical PDE solvers for the simple reason that the convergence coefficients are problem dependent (unlike the convergence rates); in particular, the convergence coefficients are proportional to some average measure of the solution derivatives. Using computed convergence coefficients and rates, one could determinine if a particular term in the error ansatz dominates the total discretzation error in a subsequent simulation (of the same problem) by combining these convergence parameters with the specific grid-spacing and time-steps of that simulation. Given results such as these for a combined convergence analysis, subsequent restricted (i.e., space-only or time-only) verification studies are possible under certain constraints on the grid-spacing and timestep.[18] Restricted verification analysis, however, would fail to capture the mixed time- and space-dependent term of the discretization error model in Eq. 1.

## Conclusions

In this study we have performed convergence analysis *simultaneously* in both space and time on a smooth problem for a Godunov scheme using Lax-Wendroff time integration. The fundamental assumption of this analysis (Eq. 1) is that the mean per-cycle error in the computed solution varies as a polynomial in the computational cell size and computational timestep, with the exponents in this expression being the convergence rates. Unlike the evaluation of convergence properties for standard spatial convergence analysis, for which, e.g., there is a closed-form expression for the convergence rate, combined space-time analysis requires the numerical solution of a set of nonlinear equations. Obtaining solutions to this set of equations, therefore, is more involved than directly obtaining the convergence results in the typical space-only or time-only convergence cases.

An application of this analysis is provided using a smooth advection problem. The results of our study demonstrate that the underlying advection algorithm is indeed second order in both space and time, including the "mixed" space-and-time discretization error rate $r + s$, at all resolutions considered. These results are in good agreement with the design characteristics of the numerical method. This combined spatio-temporal analysis provides concrete evidence supporting the claim of a verified implementation of the numerical algorithms for a smooth problem involving the linear fields of the underlying equations.

It remains to examine this method for problems that exercise the nonlinear fields of the Euler equations. Additionally, it is of interest to examine all the roots of the system of nonlinear equations that govern the convergence properties.

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$t = 0.1$ Convergence Results for $32\times32$, $64\times64$, and $128\times128$ Grids

| $\Delta t$ | $N_c$ | $\mathcal{A}\times10^2$ | $p$ | $\mathcal{B}\times10^2$ | $q$ | $\mathcal{C}\times10^2$ | $r$ | $s$ |
|---|---|---|---|---|---|---|---|---|
| 1/1600<br>1/3200<br>1/6400 | 160<br>320<br>640 | 1.00 | 1.90 | 0.67 | 1.95 | 1.00 | 0.90 | 0.90 |
| 1/3200<br>1/6400<br>1/12800 | 320<br>640<br>1280 | 1.00 | 2.00 | 0.24 | 1.89 | 1.02 | 1.00 | 1.00 |

**Table 2 Convergence quantities for the smooth advection problem calculated with $32^2$, $64^2$, and $128^2$ zones on the unit square with the indicated timesteps $\Delta t$ and number of computational cycles $N_c$. The other parameters are defined in the text.**

$t = 0.1$ Convergence Results for $64\times64$, $128\times128$, and $256\times256$ Grids

| $\Delta t$ | $N_c$ | $\mathcal{A}\times10^2$ | $p$ | $\mathcal{B}\times10^2$ | $q$ | $\mathcal{C}\times10^2$ | $r$ | $s$ |
|---|---|---|---|---|---|---|---|---|
| 1/1600<br>1/3200<br>1/6400 | 160<br>320<br>640 | 1.00 | 2.00 | 0.78 | 1.97 | 1.02 | 1.01 | 1.00 |
| 1/3200<br>1/6400<br>1/12800 | 320<br>640<br>1280 | 1.00 | 2.00 | 0.28 | 1.89 | 1.05 | 1.01 | 1.00 |

**Table 3 Convergence quantities for the smooth advection problem calculated with $64^2$, $128^2$, and $256^2$ zones on the unit square with the indicated timesteps $\Delta t$ and number of computational cycles $N_c$. The other parameters are defined in the text.**

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